

MAGIC SQUARE PATTERN HOMOLOGIES

Dave Everitt*, Fania Raczinski

30 Woodland Avenue, Melton Mowbray, LE13 1DZ, United Kingdom

*E-mail: dave.everitt@me.com; <https://daveeveritt.org/art.html>

ORCID: [0000-0002-2929-6023](https://orcid.org/0000-0002-2929-6023)

E-mail: me@fania.eu; <https://fania.eu>

ORCID: [0000-0001-5400-1031](https://orcid.org/0000-0001-5400-1031)

Abstract: *When considering magic squares (number grids with a constant sum for each row, column and diagonal), rotations and reflections are considered trivial. Our investigation uses the magic line (connecting each integer in sequence) to reveal underlying homologies and to investigate whether magic squares that share magic line patterns might also be considered trivial. We have written software (also used to create artistic creative magic line visualisations and animations) to aid this process. The mathematics is simple but magic square variations are deceptively complex and overlapping. Simplification, using homologous patterns, is the subject of this paper.*

Keywords: magic squares, combinatorics, number theory, group theory, permutation.

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1 INTRODUCTION

The 880 magic squares of order four, originally outlined in 1693 by Frénicle and proved by Friedrich Fitting (1862-1945) (see O'Connor, Robertson, 2014) were published by Suzuki (2019), which provided our base data. These are the accepted group of unique order four magic squares, excluding numerically 'trivial' rotations or reflections (D8 symmetries) from other squares. However, on considering the patterns formed by the

magic line¹ it is apparent that many of these 880, share *identical magic lines* despite differing arrangements of integers; these homologies appear in groups of two, four and eight.

This led us to look at the connection between identical magic line patterns, and the variations of integer relationships between them. Behind the ‘visual homologies’ are various shifts within the number grid that *preserve numerical order along the magic line*. Our argument is that magic line pattern homologies suggest possible new ‘trivial’ or ‘parent’ categories within the 880 unique squares. Existing ways of grouping magic squares focus on numerical tranformations e.g. row and column swapping (Heinz, 2010; Suzuki, 2019), but the magic line offers a *diagram of relationships between the numbers and their position within the grid*.

2 METHOD

We made an initial sweep of the 880, using SVG (Scalable Vector Graphics) to draw the magic line for each, so their lengths can be calculated programatically. There is a shortest and longest path through any $n \times n$ magic square grid, which can be used to form groups that share the same *length* of magic line, if not necessarily the same magic line *pattern*.

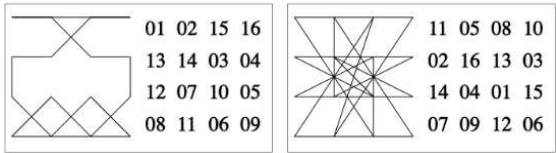


Figure 1: The shortest and longest possible order 4 magic lines (a, b).

Some magic squares have unique magic line lengths (e.g. Fig. 1a) not shared with any others – their magic line pattern could be considered ‘prime’ – but groups that share the same length of magic line are the current focus of this research. Having written a method of grouping magic lines of the same length, but without a computational method to compare their differences, we confirmed ‘homology groups’ by visual analysis, resulting in a (tentative, to be confirmed) reduced subset of 383 unique magic line patterns within the original 880 magic squares. This subset, originally created for artistic purposes (Everitt, Raczinski, 2020), contains only those with *unique magic line patterns*, providing

¹ Standard magic squares are normalised by starting with either 0 or 1; we have used 1, so the last point is n^2 where n is the order of magic square. So in an order 4 square, the magic line joins integers sequentially from 1 to 16. We use the term ‘standard magic square’ to distinguish this from other magic square sequences that may not begin with 1, contain non-contiguous numbers, or square numbers, and other variations.

a filter for deeper investigation. It also raised the question crucial to this research: Why do so many of the 880 distinct magic squares share *identical magic lines*?

Unlike existing numerical groupings that transpose integers in the grid from an initial square e.g. Suzuki's row/column switches from a fundamental set of 220 (2019), rules for this pattern-based subset create immutable magic line patterns, despite shifts in the integers. The resulting numerical transformations within these shared patterns are quite distinct, and outlined below. Incidentally, D8 symmetries of the magic line, *not* the number grid, were initially present in groups with shared magic lines drawn from the 880 set, but these are easily resolved and do not concern the 'trivial' numerical D8 symmetries of magic squares (example: Fig. 2).

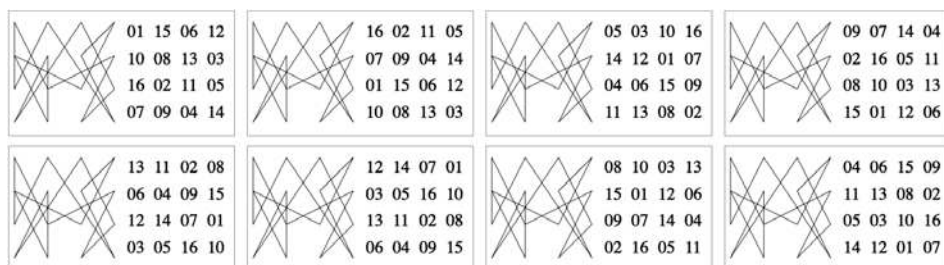


Figure 2: A group of 8 magic squares sharing a homologous magic line (a, b, c, d; e, f, g, h).

3 RESULTS

Significantly, on examining the number grids of magic squares with homologous magic line patterns, certain numerical transformations appear. The key finding is that – because the numbers are arranged along the magic line, and because the magic line is immutable i.e. does not change form – the numbers generating numerically different magic squares but sharing a magic line pattern are arranged in differing sequences that, crucially, *preserves the same relationships* between each other along magic line. A more fine-grained analysis shows that these ‘homologous pattern groups’ fall into sub-categories:

1. ‘Complementary’: where the numbers in the $n \times n$ matrix swap places from start to end i.e. for a 4×4 magic square, 1, 2, 3... exchange places with 16, 15, 14... etc. (Fig. 2a and 2b). This category is widely recognised numerically, yet not connected to the magic line (see Danielsson, 2020).
2. ‘ x -addition’: where x as certain factors of n are added to each number, subtracting n^2 where the result exceeds n^2 . The magic line retains the pattern

but the starting point and subsequent integers are shifted along the line, e.g. for 4-addition the positions of 1,2,3,4...16 become occupied by 5,6,7,8...1 (e.g. Figure 2c, 2d, 2e).

3. 'Shift-reverse': a combination of 'complementary' and 'x-addition', where the number sequence is reversed, but also 'shifted' along the grid e.g. 1,2,3,4...15, 16 is replaced by 12,11,10,9....14,13 (e.g. Figure 2f, 2g, 2h)

4 DISCUSSION

Identifying groups of magic squares that display identical magic lines suggests a new way of grouping magic squares by magic line pattern, or sequence of integers along the magic line. These groups appear as new yet fundamental sets within magic square variations, distinct from existing numerical transformations, although – in some cases – overlapping existing row/column transformations; the crucial difference being that the primacy of the magic line forms differing groups overall. They also apply to any order of magic square above order 4. Pending further investigation, this approach may also server to simplify the processing time needed to calculate the total variations for any given order.

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